SYMMETRY BREAKING of VIBRATING INTERFACES: A MECHANISM for MORPHOGENESIS

N.García and V.V.Osipov
Laboratorio de Física de Sistemas Pequeños y Nanotecnología,
Consejo Superior de Investigaciones Científicas, c/Serrano 144, 28006
Madrid, Spain
(February 2, 2008)

We show that very small-amplitude oscillations of a highly symmetric, spheric or cylindrical, interface (thin membrane) between two fluids can result in inhomogeneous instability and breaking of the interface symmetry: the frequency of the breathing vibration selects the spatial symmetry. This mechanism may govern morphogenesis.

05.65.+b, 47.20.-k, 87.10.+e

The nature of spontaneous symmetry breaking remains one of the most enigmatic questions of modern science. This problem emerges in connection with the equilibrium phase transitions, self-organization in nonequilibrium systems and many other areas in physics, chemistry and biology (see, e.g., [1]), as well as with cell fission and morphogenesis, i.e., the development and spatial differentiation of complex body structures during their growth [2].

In 1952 Turing showed that the homogeneous state of some specific chemical reactions can lose stability with regard to a spontaneous increase of perturbations of certain form [3]. Since then the chemical basis is the prevalent idea of phenomenological theory of morphogenesis (see, e.g., [1,2]). Turing's model is based on chemical or biological processes of local self-reproduction of some chemical agent (the activator) and far-ranging inhibition. As a consequence of such processes a very small increase of the activator concentration in a local region results in a global redistribution of the substance concentrations and formation of more complex structure [1,2]. However, the Turing's chemical reactions are uncommon, unique and very complex processes.

In this work we develop a new mechanism, without complexity, that breaks the symmetry by creating an instability in an oscillating interface, thin membrane, separating two different fluids. In other words, we show that if, for example, a spherical or cylindrical structure vibrates with a breathing symmetric mode for a given set of the frequencies the symmetry breaks with respect to bimodal, trimodal, pentagonal, etc. modes, i.e., the vibration frequency selects the spatial symmetry of the interface.

We consider a thin symmetric membrane, spherical or cylindrical interface, with the radius R_0 which separates two fluids with densities $\rho_1$and ρ_2 ($\rho_1 \simeq \rho_2 \simeq \rho$) respectively. Owing to Archimed's force the effective gravity acceleration operating on the internal fluid is $g = g_e(1 - \rho_1/\rho_2) << g_e$. We propose R_0 is small enough, so the condition $\gamma k_m^2/\rho = \gamma m^2/R_0^2\rho >> g$ is valid. Here γ is the surface tension and $k = m/R_0$ is the typical wave vector of the increasing deformation of the symmetric interface, $m = 1, 2, 3, ...$ This is the condition when we can neglect the gravity and consider only the effect due to the surface tension of the interface.

Let us take, at first, for definiteness, a spherically symmetric interface **S** whose radius, R, oscillates with a frequency $\omega : R = R_0 - d\cos\omega t$. From the incompressibility of the fluid it follows that its radial velocity is $v_{r0} = v_{R0}(t)R_0^2/r^2$, where $v_{R0}(t) = dR/dt = d\omega\sin\omega t$. (This means that some source, for example, a small pulsating ball has to be inside the interface.) The vortex-free motion of an ideal liquid (we consider the effect of the viscosity below) is described by the Euler and the continuity equations:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \tag{1}$$

$$\nabla^2\Phi = \nabla_r^2\Phi + \nabla_\perp^2\Phi = 0 \tag{2}$$

where Φ is the velocity potential, $\mathbf{v} = \nabla\Phi$ and $\nabla_\perp^2$ is the part of Laplacian depending only on coordinates of the surface **S**. For the undistorted spherical surface, from the symmetry of the problem, it follows that $\mathbf{v}_\perp = 0$, i.e., $\nabla_\perp^2\Phi = 0$. Then, from Eq.2, we can write that $\nabla_r^2\Phi = \nabla_r(v_{r0}) = \partial v_{r0}/\partial r + 2v_{r0}/r = 0$, in accord with $v_{r0} = v_{R0}(t)R_0^2/r^2$. In the presence of a distortion, ς, of the spherical surface **S** the interface radial velocity is $v_r = v_{R0}(t) + \partial\varsigma/\partial t$. Using this, we find from Eq.1 that near the interface

$$\frac{dv_r}{dt} = F(t) + \partial^2\varsigma/\partial t^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3}$$

$$\frac{\partial \mathbf{v}_\perp}{\partial t} = -\frac{1}{\rho}\nabla_\perp p \tag{4}$$

where $F(t) = d\omega^2 \cos\omega t$ is the acceleration of the interface and we neglect the term $(\mathbf{v}_\perp\nabla)\mathbf{v}_\perp$ in Eq. 4 by virtue of smallness of ς [4]. Owing to smallness of ς we can write the pressure near the surface as

$$p = \rho F(t)(r - R - \varsigma) + \gamma\nabla_\perp^2\varsigma + p_o(t) \tag{5}$$

Here we took into account that the pressure at the interface (when $r = R + \varsigma$) is $p = \gamma(\sigma_1 + \sigma_2) + p_o(t)$ where σ_1 and σ_2 are the principal curvatures of the interface [4]: $(\sigma_1 + \sigma_2) = \nabla_\perp^2\varsigma$ since $\nabla_\perp^2\varsigma > R_0^{-1}$. Substituting Eq.5 into Eq.4 we obtain

$$\frac{\partial \mathbf{v}_\perp}{\partial t} = F(t)\nabla_\perp\varsigma - \frac{\gamma}{\rho}\nabla_\perp^3\varsigma \text{ or } \frac{\partial}{\partial t}\nabla_\perp^2\Phi = F(t)\nabla_\perp^2\varsigma - \frac{\gamma}{\rho}\nabla_\perp^4\varsigma \tag{6}$$

We will seek solutions of the problem in the following form

$$\zeta = \sum_{m=0}^{\infty} a_m(t)S_m \text{ and } \Phi = \sum_{m=0}^{\infty} c_m(t)\Psi_m(r)S_m - v_{R0}(t)R_0^2/r \tag{7}$$

where S_m is the complete orthogonal set of eigenfunctions depending only on the coordinates of the undisturbed surface $\mathbf{S}$ and satisfying the following equation

$$(\nabla_\perp^2 + k_m^2)S_m = 0 \tag{8}$$

for $r = R_0$ and the boundary conditions corresponding to the symmetry of the problem. In the spherical case $S_m = C_{l,m}P_l^{|m|}(\cos\theta)\exp(im\varphi)$ are the spherical functions of angles φ and θ and $k_m^2 = l(l+1)R_0^{-2}$ where $m = l, l-1, ..., -l$ and $l = 0, 1, 2, ...$ Substituting Φ from Eq.7 into Eq.2, using Eq.8 and the condition $\nabla_r(v_{r0}) = 0$ cited above, we obtain the equation for $\Psi_m(r)$:

$$(\nabla_r^2 - k_m^2)\Psi_m(r) = 0 \tag{9}$$

with the boundary conditions $\nabla_r\Psi_m \to 0$ when $r \to 0$ and $\Psi_m(r) = A$ at $r = R_0$ where A is some constant which does not reveal itself in the final results. Near the interface $\nabla_r\Phi = v_r = v_{R0}(t) + \partial\varsigma/\partial t$ and so from Eq.7 it follows that $c_m(t) = da_m/dt(\nabla_r\Psi_m)^{-1}_{r=R_0}$. Substituting Φ from Eq.7 into Eq.2 and using Eq.9 and $c_m(t)$, we find that

$$\nabla_\perp^2\Phi = -\sum_{m=0}^{\infty} k_m^2\varkappa_m^{-1}S_m da_m/dt \tag{10}$$

where $\varkappa_m = [\nabla_r\Psi_m/\Psi_m(r)]_{r=R_0}$ does not depend on the constant A . Then from Eq.10 and Eq.6, we obtain

$$d^2a_m/dt^2 + [\gamma k_m^2\varkappa_m\rho^{-1} - \varkappa_m F(t)]a_m = 0. \tag{11}$$

Using $T = \omega t/2$ we can rewrite Eq.11 as

$$d^2a_m/dT^2 + (p_m - 2q_m\cos\omega t)a_m = 0, \tag{12}$$

where

$$q_m = 2\varkappa_m d \text{ and } p_m = \Omega_m^2\omega^{-2} \text{ where } \Omega_m^2 = 4k_m^2\varkappa_m\gamma\rho^{-1}. \tag{13}$$

For the spherical interface $k_mR_0 > 1$ and $\varkappa_m \simeq k_m = [l(l+1)]^{1/2}R_0^{-1}$ and so $\Omega_m^2 = 4[l(l+1)]^{3/2}R_0^{-3}\gamma\rho^{-1}$ and $q_m = 2d[l(l+1)]^{1/2}R_0^{-1}$.

These results can be extended easily to other cases. For example, when the interface have a form of a cylinder with vibrating radius, then $S_m = \cos(k_l z)\exp(im\varphi)$ and in Eq.13 $\varkappa_m \simeq k_m$ and $k_m^2 = m^2/R_0^2 + \pi^2l^2/h_0^2$ where h_0 is height of the cylinder. This vibrating cylindrical body can spontaneously distort in the axis z or with respect the azimuthal perturbations.

We emphasize that Eq.12 coincides with Eq.(2.12) of Ref. [5] to describe the Faraday's instability [6] of the plane free surface of an ideal liquid under vertical periodic vibrations. These equations differ in the values of the parameters p_m and q_m. Moreover, in contrast to the Faraday's instability when the vibrations are reduced to trivial renormalization

of the gravity, in this work we consider spherical or cylindrical oscillating interfaces when the vertical direction, axial gravity, is not distinguished from other directions. Benjamin and Ursell [5] have constructed the stability diagram for Eq.12 with respect to the universal parameters p_m and q_m using the analogy between Eq.12 and the Mathieu's equation [7]. From this diagram it follows that the instability is realized only in regions near the points $p_m = n^2$ where $n = 1, 2, 3, 4, ...$ In other words, the condition

$$\omega = \omega_{n,m} \simeq n^{-1}\Omega_m = 2n^{-1}k_m(\varkappa_m\gamma/\rho)^{1/2} \tag{14}$$

determines the resonant vibration frequencies when the symmetric interface spontaneously deforms with respect to the standing wave with the azimuthal number m. However, the greater is n, the narrower is the width $E_g^{(n)}(q_m)$ of the n-th region of the instability for given q_m [5,7]. For the widest instability region, with $n = 1$, the value $E_g^{(1)}(q_m) \simeq 2q_m$ for $q_m < 1$. It means that the instability takes place when $(1 - q_m) < p_m < (1 + q_m)$, i.e., the symmetry breaking is realized for the vibration frequency lying within the spectral range:

$$\Omega_m(1 - \varkappa_m d) < \omega < \Omega_m(1 - \varkappa_m d). \tag{15}$$

The threshold of the vibration amplitude d is limited by the fluid viscosity. For real fluid Eqs.11 and 12 include the additional terms $\gamma_m da_m(t)/dt$ and $\Gamma_m da_m(T)/dT$, respectively, where $\gamma_m = 2\nu k_m^2 C_{1m}$ and $\Gamma_m = 4\nu k_m^2 C_{1m}/\omega$ are proportional to the kinematic viscosity ν and C_m is some constant of the order of unity [8]. The threshold vibration amplitude, $d = d_t$, for the instability region with $n = 1$, can be estimated from the condition $E_g^{(1)}(q_m) > 2\Gamma_m$, i.e., $q_m = 2\varkappa_m d > \Gamma_m$. This condition follows practically from results of Refs. [8] and is obtained in [9]. Using Eq.15, this condition can be written as

$$d > d_t = 2\nu C_m k_m^2/\varkappa_m\omega \simeq \nu(\rho/\gamma\varkappa_m) \simeq \nu(\rho R_0/\gamma m). \tag{16}$$

For parameters of water $d_t \simeq 4\mu m$, i.e., the threshold vibration is a very small flutter of the interface.

We propose that the results above may be used as a basis for a simple, without complexity, mechanism to trigger the fanciful morphogenesis appearing in nature. The frequency of homogeneous interface vibrations self-selects the space symmetry. If the interface oscillates with a characteristic frequency the germ symmetry will break when its radius R_0 amounts to the quantity satisfied by Eq.15 or Eq.14 for $n > 1$. After the new symmetry appears the growth rate increases with surface curvature as is usual for many of Stefan-like problems [10].

The mathematical results reported here will be applied in a forthcoming paper to explain the morphogesis of the acetabularia, equinoderms and cell fision.

This work has been supported by the Spanish DGCIYT and by a NATO fellowship Grant.

---

[1] G. Nicolis and I. Prigogine, *Self-Organization in nonequilibrium systems,* Wiley, N.Y., London, 1977; M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65,** 851 (1993); B. S. Kerner and V. V. Osipov, *Autosolitons: A new Approach to Problems of Self-Organization and Turbulence,* Kluwer, Dordrecht, Boston, London, 1994; C. B. Muratov and V. V. Osipov, Phys. Rev. **E 54,** 4860 (1996) and Phys. Rev. **E 53,** 3101 (1996).

[2] H. Mainhardt, *Models of Biological Formation (*Academic, NY,1982*);* H. Mainhardt, *Patttern formation in Biology,* Rep. Prog. Phys. **55,** 797 (1992); B. N. Belintsev, Usp. Fiz. Nauk 141, 55 (1983) [Sov.Phys. Usp. **26,** 775 (1983)]; L. A. Segel, *Modelling Dynamic Phenomena in Molecular and Cellular Biology* (Cambridge University Press, Cambridge, U.K., 1984); G. M. Malacinski and S. V. Bryant, Eds., *Pattern Formation, A Primer in Developmental Biology* (Macmillan, NY, 1984); J. D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin,1989); B.S.Kerner and V.V.Osipov, *Structures in Different Models in Morphogenesis,* in *Self-organization, Autowaves and Structures Far from Equilibrium,* edited by V.I.Krinsky (Springer-Verlag, Berlin), p. 265-319, 1984.

[3] A. M. Turing, Phil. Trans. R. Soc. London **B 237,** 37 (1952).

[4] L. D. Landay and E. M. Lifshits, *Fluid Mechanics* (Pergamon, 1987).

[5] T. B. Benjamin and F. Ursell, Proc. R. Soc. London, Ser.A **225**, 505 (1954).

[6] M. Faraday, Phil. Trans. R. Soc. London **52**, 319 (1831).

[7] N.W.McLachlan, *Theory and application of Mathieu functions* (Oxford University Press, Oxford, 1947).

[8] E. A. Cerda and E. L. Tirapequi, J. Fluid Mech. **368,** 195 (1998). W. J. Miles, Proc. R. Soc. Lond A **297**, 459 (1967); S. T. Milner, J. Fluid Mech. **225,** 81 (1991); K. Kumar and K. M. S. Bajaj K, J. Fluid Mech. **278,** 123 (1994); K. Kumar, Proc. R. Soc. London, Ser.A **452**, 1113 (1996); K. Kumar and L. S. Tuckerman, J. Fluid Mech. **279,** 49 (1994); H. W. Müller, H. Wittmer, C. Wagner, J. Albers, and K. Knorr, Phys. Rev. Lett. **78,** 2357 (1997)..

[9] N. Garcia and V. V. Osipov, (unpublished).
[10] J. S. Langer, Revs. Mod. Phys. **52,** 1(1980); J. Chadam and P. Ortoleva, *Moving interfaces and their stability* (Gordon and Breach, N.Y., 1982); V. V. Gafiichuk, I. L. Lubashevkii and V. V. Osipov, *Dynamics of the formation of surface structures (*Naukova Dumka, Kiev, 1990).